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Abstract: We show that the Hořava theory for the completion of General Relativity at UV scales can be interpreted as a gauge fixed Tensor-Vector theory, and it can be extended to an invariant theory under the full group of four-dimensional diffeomorphisms. In this respect, although being fully relativistic, it results to be locally anisotropic in the time-like and space-like directions defined by a family of irrotational observers. We show that this theory propagates generically three degrees of freedom: two of them are related to the four-dimensional diffeomorphism invariant graviton (the metric) and one is related to a propagating scalar mode of the vector. Finally, we note that in the present formulation, matter can be consistently coupled to gravity.

Contents

1 Introduction

The main difficulty for a perturbative renormalization of General Relativity (GR) is that self-gravitational couplings are irrelevant operators at the Gaussian fixed point. Although non-trivial fixed points for the GR coupling constants might exist, corresponding to an asymptotically safe theory [1–3], they are still very difficult to find [4].

It is instructive to recall why the Gaussian fixed point makes gravity non-renormalizable. Consider the linearized gravity $g_{\mu\nu} = \eta_{\mu\nu} + \bar{g}_{\mu\nu}$, where $\eta_{\mu\nu}$ represent the flat metric. The Einstein-Hilbert action can then be schematically expanded as [5]

$$S = \frac{1}{2\kappa^2} \int d^4x \left[(\partial\bar{g})^2 + (\partial\bar{g})^2\bar{g} + \ldots \right] , \qquad (1.1)$$

where the second term is a typical self-interaction term and κ is the gravitational coupling constant of negative mass dimension, i.e. $[\kappa] = -1$. Let us now use the Gaussian fixed point at UV. For perturbative renormalization, one defines $\tilde{g} = \bar{g}/\kappa$ so that, for energies up to the cut-off of the theory, the theory can be expanded in power of κ around the free field theory (Gaussian fixed point). Expanding around the fixed point we have

$$S = \frac{1}{2} \int d^4x \left[(\partial\tilde{g})^2 + \kappa(\partial\tilde{g})^2\tilde{g} + \ldots \right] . \qquad (1.2)$$

Since κ has dimensions of length one expects that, under renormalization, $\kappa \propto 1/E$, at least for energies higher than the Planck mass [4]. This implies that the self-coupling is an irrelevant operator that spoils the renormalizability of the theory at the Gaussian fixed point [5].

The key idea for a perturbative renormalization of gravity is therefore to find non-trivial fixed points. For example, new UV fixed points appear whenever the Einstein-Hilbert action is modified by the introduction of higher order four-dimensional curvatures [6]. However, generically, in this way one introduces ghost states due to the higher order temporal

derivatives of the metric. Significant progress towards curing this bad behaviour was made by Hořava [7], whose elegant idea was to write down a theory that treats undemocratically space and time in an anisotropic way. Precursors of this idea are Lorentz violating theories [8] and, more recently, theories with ghost condensation [9].

In the Hořava theory, one hopes to obtain a higher spatial derivative theory of gravity, but canonical in time, which is power counting renormalizable (due to the change in mass dimension of the gravitational coupling) without propagating ghosts modes. To illustrate this let us suppose that one can find a modification of GR such that the action (1.1) is replaced by

$$S = \frac{1}{2\kappa^2} \int d^4x \left[-\dot{\bar{g}}^2 + \partial_i \bar{g} \partial^i \bar{g} + \alpha^{2(z-1)} \left(\partial_{i_1} \ldots \partial_{i_z} \bar{g} \right)^2 + (-\dot{\bar{g}}^2 + \partial_i \bar{g} \partial^i \bar{g})\bar{g} + \ldots \right] , \quad (1.3)$$

where we have explicitly divided the action into parts containing only time ($\dot{}$) and space (∂_i) derivatives. The constant $z \geqslant 2$ is called the anisotropy exponent by the condensed matter community, where these kind of theories were first employed. The coupling constant for the higher spatial derivative part of the action has negative mass units, i.e. $[\alpha] = -1$.

One can now show that, at the level of the Landau theory, the fixed point of (1.3) is the Lifshitz point (for an introduction of the Landau theory at the Lifshitz point see, for example, [10], whereas for a quite extensive discussion in the present context see [11]). The UV free theory related to (1.3) is

$$S = \frac{1}{2\kappa^2} \int d^4x \left[-\dot{\bar{g}}^2 + \alpha^{2(z-1)} \left(\partial_{i_1} \ldots \partial_{i_z} \bar{g} \right)^2 \right] . \quad (1.4)$$

Following the same steps as before we will now make the redefinition $\tilde{g} = \alpha^{z-1} \bar{g}/\kappa$ and choose units of time (time re-scaling) $\tilde{t} = t \, \alpha^{z-1}$. With these definitions our interacting theory becomes

$$S = \frac{1}{2} \int d^4x \left[-\dot{\tilde{g}}^2 + \lambda_1 \partial_i \tilde{g} \partial^i \tilde{g} + \left(\partial_{i_1} \ldots \partial_{i_z} \tilde{g} \right)^2 + \lambda_2(-\dot{\tilde{g}}^2 + \lambda_1 \partial_i \tilde{g} \partial^i \tilde{g})\tilde{g} + \ldots \right] , \quad (1.5)$$

where the new physical coupling constants $\lambda_1 = \alpha^{-2(z-1)}$ and $\lambda_2 = \kappa \alpha^{-(z-1)}$ have now positive mass dimensions! The dangerous irrelevant operator of GR has now been turned into a relevant operator at the Lifshitz point. This theory is then power counting renormalizable.

The innovative result of Hořava in [7, 12] was to realize that a Landau model for gravity, with critical exponent $z = 3$, could be written at the price of giving up the full diffeomorphisms invariance of GR. In this paper we will show that this requirement is actually not necessary and the theory may be written in a covariant, four-dimensional diffeomorphism invariant way.

2 Relativistic propagator at the Lifshitz point

The properties of the action (1.3) are captured by the fact that the free theory, at the Lifshitz point, has schematically the following scaling for the propagator (ignoring the tensorial structure)

$$P = -\frac{1}{\omega^2 - k_i k^i - \alpha^4 (k^i k_i)^3} , \quad (2.1)$$

where we have already specialized to the $z = 3$ case, which is particularly interesting for constructing a power counting renormalizable theory of gravity. In the UV (high energies, small length scales), the propagator is determined by the following scaling

$$P \simeq -\frac{1}{\omega^2 - \alpha^4 (k^i k_i)^3} \ . \tag{2.2}$$

The form of the propagator seems to be related to the very profound assumption that gravity is not fully covariant under space-time diffeomorphisms. Is that true? It depends. In the following we will show that the propagator (2.1) can be re-written in fully diffeomorphism invariant form. The key idea is to realize that the form of the propagator (2.1) is related to a specific choice of coordinates.

Let us suppose that we can construct a fully diffeomorphism invariant theory of gravity with UV completion, having the following propagator for the linearized graviton on a fixed background at UV scales

$$P_{\text{rel}} \simeq -\frac{1}{k^2 - \alpha^4 (\bar{h}^{\alpha\beta} k_\alpha k_\beta)^3} \ . \tag{2.3}$$

Here we have introduced

$$h_{\alpha\beta} = g_{\alpha\beta} + n_\alpha n_\beta \,, \tag{2.4}$$

which is the space-like metric orthogonal to a time-like direction n^α, i.e. $h_{\alpha\beta} n^\beta = 0$ and have also used a bar notation for the background quantities.

In order to get the propagator (2.3), the physical fields of our gravitational theory should be a metric $h_{\alpha\beta}$ parametrized by a "time" parameter and a non-dynamical form n_α. To achieve this, we consider a foliation of spacetime determined by the gradient of a scalar (χ) that can be later used to define time by fixing the choice of coordinates. In this way the local rest frame orthogonal to n^α does not "rotate" along n^α. This requirement "preserves" the spatial properties of the foliation.[1] In other words, this foliation, known as slicing of spacetime [13], defines a three-dimensional sub-Riemannian spatial manifold Σ_χ for each foliation parametrized by χ, whose spatial metric is the inverse of $h^{\alpha\beta}$. Namely, in this decomposition, the projected Christoffel symbols of the four-dimensional metric correspond to the three-dimensional Christoffel symbols related to the spatial metric. With this structure, one may projects four-dimensional tensors to the three-dimensional slices (Σ_χ) and defines compatible covariant derivatives D on Σ_χ with $h_{\alpha\beta}$ by using the projector $h^\alpha_\beta = g^{\alpha\gamma} h_{\gamma\beta}$ as follows

$$D_\gamma T^{\alpha_1 \dots \alpha_k}{}_{\beta_1 \dots \beta_\ell} = h^{\alpha_1}_{\delta_1} \cdots h^{\kappa_\ell}_{\beta_\ell} h^\sigma_\gamma \nabla_\sigma T^{\delta_1 \dots \delta_k}{}_{\kappa_1 \dots \kappa_\ell} \,. \tag{2.5}$$

The non-trivial task is now to write down an action in which n_α is non-dynamical when the renormalizable field $h_{\alpha\beta}$ is employed. Note however that, in order to have a covariant theory, n_α must be dynamical (in the equation of motion) if the spacetime metric $g_{\alpha\beta}$

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[1]Note that, any other time-like vector would introduce an extra degree of freedom changing the form of the free propagator.

is instead used [14].[2] Recapitulating, in order to restore a broken symmetry (the full diffeomorphism invariance) an extra field, the vector n^α, must be introduced into the theory. This is reminiscent of the Stuckelberg formalism to restore a U(1) gauge symmetry in a massive U(1) theory.

As already stressed before, the normalized form n_α must satisfy the conditions of zero vorticity of the space-like foliation (or equivalently the Frobenius integrability conditions [15]). Explicitly

$$\mathcal{F}_{\mu\nu} \equiv h_\mu^\alpha h_\nu^\beta \nabla_{[\alpha} n_{\beta]} = 0 . \tag{2.6}$$

The general solution of (2.6) defines the form n_α we are looking for to be

$$n_\alpha = -N\partial_\alpha\chi , \tag{2.7}$$

where the lapse N encodes the normalization condition

$$n_\alpha n^\alpha = -1 , \tag{2.8}$$

and, repeating ourselves, χ defines the foliation of the spacetime.

Note that fixing $N = f(\chi)$ overconstraints the system and corresponds to the "projectability" condition of Hořava.

If χ is now taken as a time coordinate one can use the following adapted coordinate system (ADM decomposition [16], latin indices below run from 1 to 3)

$$ds^2 = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt) . \tag{2.9}$$

We remark that the choice of coordinates has been partially fixed by requiring that $\chi = t$. The ADM-metric can however still transform under foliation preserving diffeomorphisms.

In this decomposition, the linearized gravity (which is now fixed to be transverse to n^α) is divided into the following three modes: a tensorial mode (h_{ij}), a vector mode (N^i) and a scalar mode (N). In GR only the tensorial mode propagates and, as pointed out, we require that our theory keeps this property.

Let us suppose that the theory we are after admits a flat background. By appropriately choosing the spatial coordinates in the metric (2.9), one obtains the Minkowski line-element $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$. In this gauge, we easily find $P_{\text{rel}} = P$. However, since P_{rel} is now a four-dimensional scalar, it does not change under spacetime coordinate transformations.

The original Hořava idea was instead to keep invariant the non-relativistic propagator P by effectively "freezing-out" the choice of foliation. In order to do that, in [7], only foliation preserving diffeomorphisms where allowed (the ADM gauge freedom) restricting the covariance of the gravity theory. Here instead P_{rel} is a space-time scalar and, therefore, all its properties are invariant under coordinate transformations. Of course there is still a preferred choice of (coordinate) gauge to check the renormalizability properties of the theory, namely the ADM gauge. Nevertheless, by choosing the latter, one has to remember

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[2]We thank Jose M. Martin-Garcia for this comment.

that the group of diffeomorphisms has been partially used by fixing time, contrary to what happen in General Relativity. In fact in GR, where there is no explicit dependence upon n, one can always take $\chi = t$, which just fixes the foliation. The theory we propose instead depends upon the two dynamical fields $g_{\alpha\beta}$ and n_α. In this case, the choice $\chi = t$ can be done only at the price of partially using the diffeomorphisms group! This has an interesting consequence as we shall see later.

Although diffeomorphism invariant, a propagator of the form (2.3), treats anisotropically the space-like and time-like portions of the background manifolds (the "inner" and "outer" regions of local light-cones) in compatibility with causality. We would like to stress once more that as opposed to the non-relativistic Hořava formulation (where time direction has to be kept fixed), our anisotropy of space and time foliations is again fully relativistic (i.e. invariant under diffeomorphisms).

3 The Hořava theory

To proceed, let us consider the ADM decomposition of the metric as in (2.9), where the dynamical fields h_{ij}, N and N^i have scaling mass dimensions $0, 0$ and 2, respectively. The Hořava's theory [7] with a soft violation of the detailed balance condition (represented by the last term) is

$$S = \int dt d^3x \sqrt{h} N \left\{ \frac{2}{\kappa^2} \left(K_{ij}K^{ij} - \lambda K^2 \right) - \frac{\kappa^2}{2w^4} C_{ij}C^{ij} + \frac{\kappa^2 \mu}{2w^2} \epsilon^{ijk} \mathcal{R}_{i\ell} \nabla_j \mathcal{R}^\ell{}_k \right.$$
$$\left. - \frac{\kappa^2 \mu^2}{8} \mathcal{R}_{ij} \mathcal{R}^{ij} + \frac{\kappa^2 \mu^2}{8(1-3\lambda)} \left(\frac{1-4\lambda}{4} \mathcal{R}^2 + \Lambda_W \mathcal{R} - 3\Lambda_W^2 \right) + \mu^4 \mathcal{R} \right\} , \qquad (3.1)$$

where

$$K_{ij} = \frac{1}{2N} \left(\dot{h}_{ij} - \nabla_i N_j - \nabla_j N_i \right) , \qquad (3.2)$$

is the second fundamental form (extrinsic curvature) and

$$C^{ij} = \epsilon^{ik\ell} \nabla_k \left(\mathcal{R}^j{}_\ell - \frac{1}{4} \mathcal{R} \delta^j_\ell \right) , \qquad (3.3)$$

is the Cotton tensor. Moreover, κ, λ, w are dimensionless coupling constants, μ, Λ_W are dimensionfull constants of mass dimensions $[\mu] = 1, [\Lambda_W] = 2$ and $\mathcal{R}_{ij}$ and $\mathcal{R}$ are the three dimensional Ricci and scalar curvatures related to h_{ij}. Introducing now the coordinate time $x^0 = c\,t$, c and the effective Newton's constant are given by

$$c = \frac{\kappa^2 \mu}{4} \sqrt{\frac{\Lambda_W}{1-3\lambda}} , \qquad G_N = \frac{\kappa^2}{32\pi c} . \qquad (3.4)$$

Moreover, the action (3.1) can be split into a kinetic S_{kin} and a potential term S_V according to $S = S_{\mathrm{kin}} + S_V$, where

$$S_{\mathrm{kin}} = \int dt d^3x \sqrt{h} N \left[\frac{2}{\kappa^2} \left(K_{ij}K^{ij} - \lambda K^2 \right) \right] \qquad (3.5)$$

and

$$S_V = \int dt d^3x \sqrt{h} N \left\{ -\frac{\kappa^2}{2w^4} C_{ij} C^{ij} + \frac{\kappa^2 \mu}{2w^2} \epsilon^{ijk} \mathcal{R}_{i\ell} \nabla_j \mathcal{R}^\ell{}_k - \frac{\kappa^2 \mu^2}{8} \mathcal{R}_{ij} \mathcal{R}^{ij} \right.$$
$$\left. + \frac{\kappa^2 \mu^2}{8(1-3\lambda)} \left(\frac{1-4\lambda}{4} \mathcal{R}^2 + \Lambda_W \mathcal{R} - 3\Lambda_W^2 \right) + \mu^4 \mathcal{R} \right\} . \tag{3.6}$$

This theory is invariant under the group of restricted diffeomorphisms

$$\delta x^i = \zeta^i(x,t), \qquad \delta t = f(t), \tag{3.7}$$

since the coordinates are kept adapted to n^α, i.e., in this theory, the slicing form is fixed by the choice of time parametrization.

In (3.1) it seems to exists a preferred direction (time). For these reasons the Hořava theory has been said to be non-relativistic. Indeed, the theory so formulated does not enjoy full four-dimensional covariance. This is reminiscent of a U(1) gauge theory with action

$$S = \int d^4x \sqrt{-g} \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2a} (\partial_\mu A^\mu)^2 \right) , \tag{3.8}$$

which is invariant under the "restricted" U(1) symmetry $A_\mu \to A_\mu + \partial_\mu \theta$, where the gauge parameter θ satisfies $\nabla^2 \theta = 0$. The full U(1) symmetry with unrestricted gauge parameter seems to be broken but this is solely due to the fact that the gauge has been fixed.

Similarly, one may wonder if the restricted symmetry of (3.1) is really some fundamental property of the theory or just a choice of gauge for the full group of four-dimensional diffeomorphism. If this is the case, then, one should be able to rewrite Hořava's theory in a fully four-dimensional covariant way. This is the task of the present work.

4 A relativistic interpretation of the Hořava theory

Our aim here is to write the action (3.1) in terms of four-dimensional scalars where, for the sake of covariance, both the spacetime metric and n^α are dynamical [14]. This requirement explicitly preserves the full four-dimensional group of diffeomorphisms.

A basic geometric object entering into the action (3.1) is the extrinsic curvature K_{ij}. The extrinsic curvature associated with the particular foliation chosen, is the variation $h_{\alpha\beta}$ along the temporal direction n^α. The extrinsic curvature therefore encodes the "time" derivative of the propagating field $h_{\alpha\beta}$ and is in fact a four-dimensional object. In a four-dimensional notation, it is written as

$$K_{\alpha\beta} = \frac{1}{2} \pounds_n h_{\alpha\beta}, \tag{4.1}$$

where $\pounds_n$ is the Lie-derivative along n.

The kinetic term of the theory can therefore be written as

$$S_{\text{kin}} = \int d^4x \sqrt{-g} \left[\frac{2}{\kappa^2} \left(K_{\alpha\beta} K^{\alpha\beta} - \lambda K^2 \right) \right], \tag{4.2}$$

where λ is a constant and $K = K_\alpha^\alpha$. We remark here that the action (4.2), as it is written in terms of four-dimensional scalars, is fully diffeomorphism invariant for *any* λ contrary of what claimed in [7].

To further define the theory we need to include "potential" terms which should only depend on spatial derivatives of the metric. We realized that if n^α is irrotational, the space related to $h^{\alpha\beta}$ is a sub Riemannian space-like manifold. Therefore all curvatures computed on the foliation only depends on spatial derivatives. Nevertheless, three dimensional curvatures ($\mathcal{R}_{\alpha\beta\gamma\delta}$), i.e. curvatures of the space-like foliations, are four dimensional tensors related to the four dimensional curvatures and extrinsic curvature by the Gauss equation

$$\mathcal{R}_{\lambda\mu\nu\rho} = R_{\alpha\beta\gamma\delta}h_\lambda^\alpha h_\mu^\beta h_\nu^\gamma h_\rho^\delta + 2K_{\mu[\nu}K_{\rho]\lambda} \; . \tag{4.3}$$

$\mathcal{R}_{\lambda\mu\nu\rho}$ contains two spatial derivatives of the spatial metric. The covariant derivatives appearing in (3.1) may be promoted to four-dimensional ones by using

$$h_\mu^\alpha h_\nu^\beta h_\delta^\gamma \nabla_\alpha \mathcal{R}_{\beta\gamma} = D_\mu \mathcal{R}_{\nu\delta} \; , \tag{4.4}$$

where D is the spatial covariant derivative defined by taking the four-dimensional derivative and projecting all indices with h into the spatial foliation [13], according to (2.5). Due to the invariance of the three dimensional space, this derivative effectively only operates on the sub-Riemannian manifold. In addition, the Cotton tensor (3.3) is written in a four-dimensional notation as

$$C^{\mu\nu} = \eta^{\mu\alpha\beta} D_\alpha \left[\mathcal{R}^\nu_{\;\beta} - \frac{1}{4}\mathcal{R}\delta_\beta^\nu \right] \; , \tag{4.5}$$

where $\eta^{\mu\alpha\beta}$ is the three-dimensional volume element defined as $\eta^{\mu\alpha\beta} \equiv \eta^{\mu\alpha\beta\delta}n_\delta$ with the property $D_\alpha\eta_{\mu\nu\rho} = 0$ [13].

Until now we have always assumed that n^α is a normalized zero vorticity vector. To enforce that we introduce the action

$$S_{\text{norm}} \sim \int d^4x\sqrt{-g} \left[B^{\alpha\beta}\mathcal{F}_{\alpha\beta} + M^{\alpha\beta\mu\nu}B_{\alpha\beta}B_{\mu\nu} + \rho(n^\alpha n_\alpha + 1) \right] \; , \tag{4.6}$$

where ρ, $B_{\alpha\beta}$ and $M_{\alpha\beta\mu\nu}$ are Lagrange multipliers. Varying ρ enforces the normalization condition (2.8) on n, whereas varying the other two multipliers enforces the Frobenius condition (2.6) and $B_{\alpha\beta} = 0$. The only possible degree of freedom in n_α is obviously N as, by using the diffeomorphism invariance of the theory, one can always fix time to be $t = \chi$. The fact that there is only one degree of freedom in n_α is clear by just noticing that the irrotational constraint reduce the number of degrees of freedom from four to two.[3] The normalization condition finally leaves only one degree of freedom for n_α.

To recover GR in the IR (low energies, large length scales in the adapted coordinates to n_α), we can now add a potential $V(\mathcal{R}_{\alpha\beta})$. This potential should be taken to be at most quadratic in the three dimensional Ricci curvatures so that it does not affect the UV

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[3] The antisymmetric tensor $\mathcal{F}_{\alpha\beta}$ obeys the four conditions $\mathcal{F}_{\alpha\beta}n^\beta = 0$. That leaves $6-4 = 2$ independent components which, from the specific form of $\mathcal{F}_{\alpha\beta}$ in (2.6), are two of the components of n_α.

(high energies, small length scales in the adapted coordinates to n_α) behavior dominated by the term quadratic in the Cotton tensor. In particular, by adding the term (4.6) to the action, one can extend the original non-relativistic detailed balance model of Hořava (or the simpler model of [17]) to be invariant under the full diffeomorphism group. A four-dimensional covariant theory with the required UV properties, compatible with the Hořava theory, is therefore

$$
\begin{aligned}
S_{\mathrm{g}} = \int d^4x \sqrt{-g} \Big\{ & a_0 \mathcal{R} + a_1 K_{\alpha\beta} K^{\alpha\beta} + a_2 K^2 + a_3 \mathcal{R}_{\alpha\beta} \mathcal{R}^{\alpha\beta} \\
& -a_4 C_{\mu\nu} C^{\mu\nu} + a_5 C_{\mu\nu} \mathcal{R}^{\mu\nu} + a_6 \mathcal{R}^2 + 2\Lambda + B^{\alpha\beta} \mathcal{F}_{\alpha\beta} + M^{\alpha\beta\mu\nu} B_{\alpha\beta} B_{\mu\nu} + \\
& + \rho(n^\alpha n_\alpha + 1) \Big\},
\end{aligned}
\tag{4.7}
$$

where a_i are parameters. In this form of the action the independent fields are $h^{\alpha\beta}, n^\alpha, \rho, B_{\mu\nu}$ and $M_{\alpha\beta\gamma\delta}$. Note that, to define the spatial metric $h_{\alpha\beta}$, only the normalization condition is needed which has been indeed introduced as a constraint for the variational problem. The removal of the Frobenius conditions will only define a new theory with more degree of freedom than the Hořava theory. The Frobenius conditions define therefore the anisotropic theory with minimal degrees of freedom by restricting the possible solutions for n_α.

The theory (4.7) differs from GR mainly by the introduction of a new vectorial degree of freedom n^α which treats undemocratically the spaces parallel and orthogonal to it. In any case, as already stressed before, this new degree of freedom does not break the covariance of the theory, i.e. the action (4.7) is invariant under the full group of diffeomorphisms

$$
\delta x^\mu = \zeta^\mu(x^\alpha), \quad \delta g_{\mu\nu} = \nabla_\mu \zeta_\nu + \nabla_\nu \zeta_\mu, \quad \delta n_\alpha = \zeta^\mu \nabla_\mu n_\alpha + n_\mu \nabla_\alpha \zeta^\mu,
\tag{4.8}
$$

as compared to (3.7).

We stress that both $g_{\alpha\beta}$ and n_α in (4.7) are dynamical in compatibility with covariance [14]. However, by field redefining $g_{\alpha\beta} = h_{\alpha\beta} - n_\alpha n_\beta$, n_α is no longer dynamical. Therefore, n^α (and in particular N) is a Lagrange multiplier for the field theory of $h_{\alpha\beta}$. Had the theory been constructed from a rotational vector, say u^α, then it would have had an extra vectorial degree of freedom.

We have finally accomplished our goal. The action (4.7) has indeed two time derivatives and higher space derivatives in the adapted coordinate frame on n_α, the ADM observer. However, in a boosted frame, higher time derivatives will be induced by space derivatives leading presumably to acausal propagation and instabilities. Nevertheless, although locally any boosted observer is as good as any other, in this theory it is not the case globally. In fact, for the privileged ADM observer there is a well posed Cauchy problem with two initial data that gives rise to physical solutions specified by appropriate boundary conditions at spatial infinity. Of course, the boundary conditions at the ADM spatial infinity correspond to very special boundary conditions for a boosted observer.[4] A non-ADM observer will probe solutions with incredibly fine tuned initial conditions such that to cancel acausal propagation and instabilities. However, the fine tuning is only apparent as the correct boundary conditions can only be imposed globally form an ADM observer. Thus, although

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[4]CG thank Oriol Pujolas for pointing this out to him.

the boosted ADM observer may account higher time derivatives in his effective action, his boundary conditions (the boosted ADM ones) will kill all the unphysical modes related to the higher time derivative solutions. In other words, boosting cannot turn normal fields into ghosts.

One may finally also check that all couplings of our theory at the Lifshitz point are related to relevant operators. Moreover, this theory can be expanded on a flat background which is a vacuum solution. In this sense, Lorentz invariance is preserved. The precise correspondence with Hořava's Lagrangian (3.1) is derived by fixing the ADM gauge and specifying the parameters to be

$$a_0 = \frac{\kappa^2 \mu^2}{8(1-3\lambda)c}\Lambda_W, \quad a_1 = \frac{2c}{\kappa^2}, \quad a_2 = -\frac{2c\lambda}{\kappa^2}, \quad a_3 = -\frac{\kappa^2\mu^2}{8c}, \quad a_4 = \frac{\kappa^2}{2w^4 c}$$

$$a_5 = \frac{\kappa^2\mu}{2w^2 c}, \qquad\qquad a_6 = \frac{\kappa^2\mu^2}{8c}\frac{1-4\lambda}{(1-3\lambda)}, \quad \Lambda = -\frac{3\kappa^2\mu^2}{16(1-3\lambda)c}\Lambda_W^2, \qquad (4.9)$$

in order to obtain the terms of the action arising from the detailed balance condition. Note that the scaling anisotropy between space and time may be reintroduced by writing $dx_0 = cdt$. To this action however a soft detailed balance breaking term must be added in order to have the correct IR GR limit of the theory [17–20] (for a detailed account of various possibilities see [21]). We will then consider the relativistic extension of the minimal theory as in [17].

The simplest covariant Lagrangian which is diffeomorphism invariant, power counting renormalizable at the Lifshitz fixed point and with the correct IR behaviour is then

$$S = \int d^4x \sqrt{-g} \Big\{ \frac{2}{\kappa^2}\left(K_{\alpha\beta}K^{\alpha\beta} - \lambda K^2\right) - \frac{\kappa^2}{2w^4}C_{\alpha\beta}C^{\alpha\beta} + \frac{\kappa^2\mu}{2w^2}\eta^{\alpha\beta\gamma}\mathcal{R}_{\alpha\epsilon}D_\beta\mathcal{R}^\epsilon{}_\gamma$$
$$- \frac{\kappa^2\mu^2}{8}\mathcal{R}_{\alpha\beta}\mathcal{R}^{\alpha\beta} + \frac{\kappa^2\mu^2}{8(1-3\lambda)}\frac{1-4\lambda}{4}\mathcal{R}^2 + \mu^4\mathcal{R} + B^{\alpha\beta}\mathcal{F}_{\alpha\beta} + M^{\alpha\beta\mu\nu}B_{\alpha\beta}B_{\mu\nu} +$$
$$+ \rho(n^\alpha n_\alpha + 1) \Big\}. \qquad (4.10)$$

Let us now count the propagating degrees of freedom of the graviton $h_{\alpha\beta}$. To do that it is simpler to re-write the theory in terms of the two fields $g_{\alpha\beta}$ and n_α. Since the theory is fully diffeomorphism invariant the tensor $g_{\alpha\beta}$ has only two propagating degrees of freedom and, as we have already stressed, n_α has only one degree of freedom. By conservation of the number of degrees of freedom under field redefinition one than discover that the propagating graviton $h_{\alpha\beta}$ contains three degrees of freedom. One might now ask whether the extra degree of freedom with respect to the usual graviton propagation in GR might go away for certain choice of parameters. Generically this is not the case.[5] In fact, in order to remove the extra degree of freedom in the system, the action should not explicitly depend upon n. This would be only in general possible if the action was written in terms of four-dimensional curvatures. In the $\lambda = 1$ case, the extra propagating mode should freeze at the IR as the theory tends to GR.

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[5]Note however that in specific cases the number of propagating degrees of freedom can be lower as we shall soon discuss.

Matter can now be consistently added in a diffeomorphism invariant way to the theory (for issues related to adding matter to the Hořava theory see [20, 22]), by minimally coupling it not only to the dynamical field $h_{\alpha\beta}$, but also to the combination $g_{\alpha\beta} = h_{\alpha\beta} - n_\alpha n_\beta$. Schematically

$$S = \int d^4x \sqrt{-g} \left[R + L_{\text{grav}} + L_{\text{m}} \right] , \tag{4.11}$$

where L_{m} is the matter Lagrangian and $L_{\text{grav}} = (\lambda - 1)K^2 - \tilde{V}(\mathcal{R}_{\alpha\beta}, D_\mu \mathcal{R}_{\alpha\beta})$.

Since this theory is invariant under four-dimensional diffeomorphisms one has the following Bianchi-type identities

$$\nabla_\alpha T_{\text{grav}}^{\alpha\beta} = 0 , \qquad \nabla_\alpha T_{\text{m}}^{\alpha\beta} = 0 , \tag{4.12}$$

where $T_{\text{grav,m}}$ are respectively the energy momentum tensors of L_{grav} and L_{m}. The conservation of the energy momentum tensor for matter alone is equivalent to the field equations of matter.

Let us suppose that the full set of diffeomorphisms for the gravity sector was not allowed, as in the Hořava theory. The field equations of motion for the matter would still imply the conservation of the energy momentum tensor (if the matter Lagrangian is kept to be a four dimensional scalar). Then, the gravity equations would require $\nabla_\alpha T_{\text{grav}}^{\alpha\beta} = 0$ (the Einstein tensor is conserved identically). This is equivalent to re-introducing the full diffeomorphism group of transformations (for a lucid explanation of this equivalence see [25]).

A related issue here concerns the "potentially" strong coupling problem reported in [20]. Although in the analysis of [20] care has been given to restore the full four-dimensional diffeomorphism invariance in the gravity sector (so that $\nabla_\alpha T_{\text{grav}}^{\alpha\beta} = 0$ holds), the matter sector is not invariant under four-dimensional diffeomorphisms making the reported strong coupling problem of Hořava gravity questionable. In our approach, the coupling of the covariant theory (4.7) to a four-dimensional diffeomorphism invariant matter sector is straightforward. In this respect, the issue of the strong coupling problem discussed in [20] could consistently be analyzed (see also [23]). This is however beyond the scope of the present work.

5   An example: cosmological perturbations

Our point is that the Hořava theory should be embedded into a Tensor-Vector theory where gravity propagates with the field $h_{\alpha\beta}$. By explicitly using the ADM decomposition the constraint (4.6) is automatically satisfied and one recover the action of [7]. However, one has to now remember that this theory comes from a fully diffeomorphism invariant action where the gauge parameter ζ^μ in (4.8) is partially fixed such that $\chi = t$. This has important consequences for example in cosmological perturbations as we shall now show.

Suppose one wishes to study scalar perturbations at the linearized level. As we have pointed out several times, the dynamical covariant system is formed by gravity $g_{\alpha\beta}$ and the field n_α. Using the gauge choice $\chi = t$, scalar perturbations in a Friedmann-Robertson-Walker background

$$ds^2 = a^2(t) \left[-dt^2 + \delta_{ij} dx^i dx^j \right] , \tag{5.1}$$

might be generically parametrized as [26]

$$ds^2 = a^2 \left[(-1 - 2A)dt^2 + 2\partial_i B dt dx^i + ((1 - 2\psi)\delta_{ij} + \nabla_i \nabla_j E) \, dx^i dx^j \right] \,. \qquad (5.2)$$

If one wishes to keep the same foliation of the background metric, one has to fix in (4.8) $\zeta^0 = \zeta^0(t)$ [26]. This partial gauge fixing still leaves the relativistic theory invariant under the smaller group of foliation preserving diffeomorphisms, as in the Hořava theory [7].

Let us now parametrize the diffeomorphisms in the usual way [26]

$$\delta t = \zeta^0(x^\mu) \,, \quad \delta x^i = \nabla^i \beta(x^\mu) + v^i(x^\mu) \,, \quad \partial_i v^i = 0 \,, \qquad (5.3)$$

then we have the following transformations for the scalar modes:

$$
\begin{aligned}
A &\rightarrow A - \dot{\zeta}^0 - \frac{\dot{a}}{a}\zeta^0 \,, \\
B &\rightarrow B + \zeta^0 + \dot{\beta} \,, \\
\psi &\rightarrow \psi - \frac{1}{3}\nabla^2\beta + \frac{\dot{a}}{a}\zeta^0 \,, \\
E &\rightarrow E + 2\beta \,.
\end{aligned}
\qquad (5.4)
$$

Fixing ζ^0 to preserve the foliation, leaves only one degree of freedom (β) in (5.4) in contradiction with GR in which the degrees of freedom are two (ζ^0, β). One can then easily see that three of the graviton components cannot be set to zero. Thus, in principle gravity generically contains three modes. Whether the three modes are all propagating or not depends really on the background (and the perturbative level). In fact in [24], in presence of matter, only one propagating linear mode has been found. Also in GR a similar thing happens. There, on a cosmological background, the physical modes are two but only one of them, or none in absence of a scalar source, propagates.

6   Conclusions

Theories that extend GR with additional vectorial degrees of freedom, but nevertheless invariant under the full group of diffeomorphisms, are not new. For example, the Bekenstein TeVeS theory of gravity [29] and the Einstein-Aether theory [30] or the "preferred frame" gravity theories of Jacobson and Mattingly [31] are of this kind. Related to our present context, one may also write down a covariant theory which couples non-minimally to a specific (dynamical) perfect fluid stress-tensor with curvatures. In this case, one might obtain a higher spatial derivative theory for the graviton [32]. This theory however introduces many extra degrees of freedom into the purely gravitational action and it is also not clear whether the GR limit at IR is obtained.

In this paper we showed that the Hořava theory for a power counting renormalizable quantum gravity, tending to GR at the IR, is nothing else but a gauge fixed diffeomorphism invariant Tensor-Vector theory. Indeed, the full diffeomorphism invariance of the Hořava theory can be restored by introducing a Lagrange multiplier which normalizes the irrotational time-like vector field n^α defining a space-like foliation of the spacetime manifold. This, seems also to resolve potential problems related to the canonical formulation of the Hořava theory [33].

As seen from the four-dimensional metric ($g_{\alpha\beta}$) perspective, the theory generically propagates two degrees of freedom from the metric, as it should happen for a diffeomorphism invariant theory of gravity, and one degree of freedom from n_α. However, this theory can also be studied by using the fields $h_{\alpha\beta}(= g_{\alpha\beta} + n_\alpha n_\beta)$ and n_α. In this base only $h_{\alpha\beta}$ propagates encoding all degrees of freedom and results in a power counting renormalizable field theory. Its orthogonal projection to n_α, defines an invariant three-dimensional Riemannian sub-manifold parametrized by the time-like direction n^α. Therefore, n^α only acts as a "Lagrangian multiplier" for the true degrees of freedom in $h_{\alpha\beta}$.

Fixing the spacetime metric to be in the ADM form as in [7] partially fixes the gauge freedom of the theory by requiring $\chi = t$, where $n_\alpha \equiv -N\partial_\alpha\chi$, restricting the diffeomorphisms invariance of the theory to be only foliation preserving.

Although the theory we presented is fully diffeomorphism invariant, it treats anisotropically the space-like and the time-like sub-spaces delineated by the local light cones of n. This might seems to violate Lorentz invariance locally. However, as the vacuum solution of the theory is still Minkowski, the linearized graviton Lagrangian is still invariant under boosts, although the graviton dispersion relations are different from the usual GR one at UV scales.

Let us now instead start from the Hořava theory where n is an external field and where only the restricted group of diffeomorphisms (3.7) is allowed. In this case if matter is taken to be standard, i.e. it is described by a four dimensional scalar Lagrangian, general covariance would imply that the divergence of the total energy-momentum tensor of all geometrical quantities should vanishes when the matter field equations are satisfied. This is a very strict physical constraint necessary for the consistency of a (diffeomorphisms invariant) matter-gravity system. If a preferred frame for the gravity sector is then introduced and general covariance is abandoned, the conservation of the matter energy momentum tensor would nevertheless reintroduce the full diffeomorphism group as a constraint equation for gravity. Of course this would not be the case if the matter Lagrangian is not described by a four dimensional scalar.

The above are reminiscent of a case described in [27, 28]. In this work a modified gravity of the form

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left(R - \frac{1}{2} u_\mu K^\mu \right) , \qquad (6.1)$$

was considered, where the vector $u^\mu = (\frac{1}{m}, 0, 0, 0)$ is a fixed background vector and K^μ is the gravitational Chern-Simon current whose divergence is the Chern-Pontryagin density ($\partial_\mu K^\mu = \frac{1}{2}{}^*\!R\,R$). The existence of the fixed vector u^μ apparently violates four-dimensional diffeomorphism invariance and Lorentz invariance. However, as it has been shown in [28], the theory is diffeomorphism invariant and there are no Lorentz violating effects. In fact, one may start with a diffeomorphism invariant theory with a varying u^μ. Then diffeomorphism invariance may be used to fix a particular gauge for u^μ. This choice seems to restrict the diffeomorphism group but this is because we consume partially the diffeomorphism group to fix a particular frame.

Thus, in general, there are cases where what appears to be a symmetry violation is just due to a gauge choice. In the Hořava theory the apparent violation of four-dimensional covariance is just because it is written in a specific gauge, specified by the ADM frame. In turn, this can be used because of the four-dimensional covariance of the theory (and the corresponding constraints). Thus, the Hořava's theory coupled to standard matter is a particular case of the general class of a relativistic, fully covariant action (4.7). If this were not the case, then there would be problems with the conservation of the energy-momentum tensor for matter covariant Lagrangians. The reason is that, apart form potential inconsistencies, violation of covariance in the gravity sector would be transmitted to the matter sector by graviton loops leading to ill defined matter theories. We conclude by stressing that our covariant theory allows consistent coupling of gravity with matter in contrast with the original Hořava theory [7].

Acknowledgments

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Erratum

The definition (4.3) is valid only if the Frobenius conditions (2.6) are satisfied. This is due to the fact that the Frobenius conditions are non-holonomic constraints. In this case, the definition of n_α in (2.7) together with the normalization (2.8), i.e.,

$$n_\alpha = -\frac{\partial_\alpha \chi}{\sqrt{-\partial_\beta \chi \partial^\beta \chi}}\,, \tag{A.1}$$

must be used in the action (4.7). The constraints (4.6) are then automatically satisfied and the theory so written is explicitly a tensor-scalar theory, where the scalar χ represents the degree of freedom of n_α.

In summary, once (A.1) is directly used in (4.7), $g^{\alpha\beta}$ and χ become the independent fields of the variational problem for the theory (4.7), in compatibility with [34].

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